Polynomial Associative Algebras for Quantum Superintegrable Systems with a Third Order Integral of Motion

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We consider a superintegrable Hamiltonian system in a two-dimensional space with a scalar potential that allows one quadratic and one cubic integral of motion. We construct the most general associative cubic algebra and we present specific realizations. We use them to calculate the energy spectrum. All classical and quantum superintegrable potentials separable in cartesian coordinates with a third order integral are known. The general formalism is applied to one of the quantum potentials.

1 Introduction

The purpose of this article is to study the algebra of integrals of motion of a certain class of quantum superintegrable systems allowing a second and a third order integral of motion. We will consider a cubic associative algebra and we will study its algebraic realization. A systematic search for superintegrable systems in classical and quantum mechanics was started some time ago[8,10,18]. The study of superintegrable system with a third order integral is more recent. All classical and quantum superintegrable potentials in E(2) that separate in cartesian coordinate and allow a third order integral were found by S.Gravel[13]. In this article we will be interested in particular in one new potentials found in Ref [13] that was studied by the dressing chain method in Ref [14].
It is well known that in quantum mechanics the operators commuting with the Hamiltonian, form an o(4) algebra for the hydrogen atom[1] and a u(3) algebra for the harmonic oscillator[15]. This type of symmetry is called a dynamical or hidden symmetry and can be used give a complete description of the quantum mechanical system. The symmetry determines all the quantum numbers, the degeneracy of the energy levels and the

energy spectrum[9,15].
Many examples of polynomial algebras have been used in different branch of physics. C.Daskaloyannis studied the quadratic Poisson algebras of two-dimensional classical superintegrable systems and quadratic associative algebras of quantum superintegrable systems[2,3,4]. He shows how the quadratic associative algebras provide a method to obtain the energy spectrum. He uses realizations in terms of a deformed oscillator algebra[5]. We will follow an analogous approach for the study of cases with third order integrals.
In an earlier article[17] we considered cubic Poisson algebras for classical potentials and applied the theory to the 8 potentials separating in cartesian coordinates and allowing a third order integral. The purpose of this article is to study cubic associative algebras. We find the realization of these polynomial associative algebras in terms of a parafermionic algebra. From this we find Fock type representations and the energy spectrum. We reduce this problem to the problem of solving two algebraic equations. This article provides another example of the fact that it is very useful to consider not only Lie algebras in the study of quantum systems but also polynomial algebras.

2 Cubic Associative algebras and their algebraic realizations

We consider a quantum superintegrable system with a quadratic Hamiltonian and one second order and one third order integral of motion.

$$\begin{gathered}
H = a(q_1,q_2)P_1^2 + 2b(q_1,q_2)P_1P_2 + c(q_1,q_2)P_2^2 + V(q_1,q_2) \\
A = A(q_1,q_2,P_1,P_2) = d(q_1,q_2)P_1^2 + 2e(q_1,q_2)P_1P_2 + \\
f(q_1,q_2)P_2^2 + g(q_1,q_2)P_1 + h(q_1,q_2)P_2 + Q(q_1,q_2) \qquad (2.1) \\
B = B(q_1,q_2,P_1,P_2) = u(q_1,q_2)P_1^3 + 3v(q_1,q_2)P_1^2P_2 + 3w(q_1,q_2)P_1P_2^2 \\
+x(q_1,q_2)P_2^3 + j(q_1,q_2)P_1^2 + 2k(q_1,q_2)P_1P_2 + l(q_1,q_2)P_2^2 + \\
m(q_1,q_2)P_1 + n(q_1,q_2)P_2 + S(q_1,q_2)
\end{gathered}$$

with

$$P_1 = -i\hbar\partial_1, P_2 = -i\hbar\partial_2 \qquad (2.2)$$

$$[H, A] = [H, B] = 0 \tag{2.3}$$

We assume that our integrals close in a polynomial algebra :

$$\begin{aligned}
[A, B] &= C \\
[A, C] &= \alpha A^2 + \beta\{A, B\} + \gamma A + \delta B + \epsilon \\
[B, C] &= \mu A^3 + \nu A^2 + \rho B^2 + \sigma\{A, B\} + \xi A + \eta B + \zeta
\end{aligned} \tag{2.4}$$

where $\{\}$ denotes an anticommutator.

The Jacobi identity $[A, [B, C]] = [B, [A, C]]$ implies $\rho = -\beta$, $\sigma = -\alpha$ and $\eta = -\gamma$.

$$\begin{aligned}
[A, B] &= C \\
[A, C] &= \alpha A^2 + \beta\{A, B\} + \gamma A + \delta B + \epsilon \\
[B, C] &= \mu A^3 + \nu A^2 - \beta B^2 - \alpha\{A, B\} + \xi A - \gamma B + \zeta \quad .
\end{aligned} \tag{2.5}$$

The Casimir operator of a polynomial algebra is an operator that commutes with all elements of the associative algebra. The Casimir operator satisfies :

$$[K, A] = [K, B] = [K, C] = 0 \tag{2.6}$$

and this implies

$$\begin{gathered}
K = C^2 - \alpha\{A^2, B\} - \beta\{A, B^2\} + (\alpha\beta - \gamma)\{A, B\} + (\beta^2 - \delta)B^2 \\
(+\beta\gamma - 2\epsilon)B + \frac{\mu}{2}A^4 + \frac{2}{3}(\nu + \mu\beta)A^3 + (-\frac{1}{6}\mu\beta^2 + \frac{\beta\nu}{3} + \frac{\delta\mu}{2} + \alpha^2 + \xi)A^2 \\
+(-\frac{1}{6}\mu\beta\delta + \frac{\delta\nu}{3} + \alpha\gamma + 2\zeta)A
\end{gathered} \tag{2.7}$$

We construct a realization of the cubic associative algebra by means of the deformed oscillator technique. We use a deformed oscillator algebra $\{b^t, b, N\}$ which satisfies the relation

$$[N, b^t] = b^t, [N, b] = -b, b^t b = \Phi(N), b b^t = \Phi(N + 1) \tag{2.8}$$

We request that the "structure function" $\Phi(x)$ should be a real function that satifies the boundary condition $\Phi(0) = 0$, with $\Phi(x) > 0$ for $x > 0$. These constraints imply the existence of a Fock type representation of the deformed oscillator algebra[4,5]. There is a Fock basis $|n>$, n=0,1,2... satisfying

$$N|n >= n|n >, \quad b^t|n >= \sqrt{\Phi(N+1)}|n+1> \tag{2.9}$$

$$b|0 >= 0, \quad b|n >= \sqrt{\Phi(N)}|n-1> \tag{2.10}$$

We consider the case of a nilpotent deformed oscillator algebra, i.e., there should be an a integer p such that,

$$b^{p+1} = 0, (b^t)^{p+1} = 0 \tag{2.11}$$

These relations imply that we have

$$\Phi(p+1) = 0 \tag{2.12}$$

In this case we have a finite-dimensional representation of dimension p+1.

Let us show that there is a realization of the form :

$$A = A(N), B = b(N) + b^t\rho(N) + \rho(N)b \tag{2.13}$$

The functions A(N) , b(N) et $\rho(N)$ will be determined by the cubic associative algebra, in particular the first and second relation. We use the commutation relation of the cubic associative algebra to obtain

$$[A, B] = C$$

$$[A, B] = b^t \triangle A(N)\rho(N) - \rho(N) \triangle A(N)b \tag{2.14}$$

$$\triangle A(N) = A(N+1) - A(N)$$

$$[A, C] = \alpha A^2 + \beta\{A, B\} + \gamma A + \delta B + \epsilon$$

$$= b^t(\gamma(A(N+1) + A(N)) + \delta)\rho(N)+ \tag{2.15}$$

$$\rho(N)(\gamma(A(N+1)+A(N))+\delta)b+\alpha A^2(N)+2\beta A(N)b(N)$$
$$+\gamma A(N)+\delta b(N)+\epsilon$$

using

$$[A,C]=b^t\triangle A(N)\rho(N)\triangle A(N)+\triangle A(N)\rho(N)\triangle A(N)b$$
$$=b^t\triangle A(N)^2\rho(N)+\rho(N)\triangle A(N)^2 b \tag{2.16}$$

we obtain two equations that allow us to determine A(N) and b(N).

$$\triangle A(N)^2=\gamma(A(N+1)+A(N))+\epsilon \tag{2.17}$$
$$\alpha A(N)^2+2\gamma A(N)+b(N)+\delta A(N)+\epsilon b(N)+\xi=0$$

We shall distinguish two cases.

Case 1 $\beta\neq 0$

$$A(N)=\frac{\beta}{2}((N+u)^2-\frac{1}{4}-\frac{\delta}{\beta^2}) \tag{2.18}$$
$$b(N)=\frac{\alpha}{4}((N+u)^2-\frac{1}{4})+\frac{\alpha\delta-\gamma\beta}{2\beta^2}$$
$$-\frac{\alpha\delta^2-2\gamma\delta\beta+4\beta^2\epsilon}{4\beta^4}\frac{1}{(N+u)^2-\frac{1}{4}}$$

The constant u will be determined below using the fact that we require that the deformed oscillator algebras should be nilpotent. The last equation of the associative cubic algebra contains the cubic term and is

$$[B,C]=\mu A^3+\nu A^2-\beta B^2-\alpha\{A,B\}+\xi A-\gamma B+\zeta \tag{2.19}$$

We obtain the equation,

$$2\Phi(N+1)(\triangle A(N)+\frac{\gamma}{2})\rho(N)-2\Phi(N)(\triangle A(N-1)-\frac{\gamma}{2})\rho(N-1) \tag{2.20}$$
$$=\mu A(N)^3+\nu A(N)^2-\beta b(N)^2-2\alpha A(N)b(N)+\xi A(N)-\gamma b(N)+\zeta$$

The Casimir operator is now realized as

$$K = \Phi(N+1)(\beta^2 - \delta - 2\beta A(N) - \triangle A(N)^2)\rho(N) \tag{2.21}$$
$$+\Phi(N)(\beta^2 - \delta - 2\beta A(N) - \triangle A(N-1)^2)\rho(N-1) - 2\alpha A(N)^2 b(N)$$
$$+(\beta^2 - \delta - 2\beta A(N))b(N)^2 + 2(\alpha\beta - \gamma)A(N)b(N) + (\beta\gamma - 2\epsilon)b(N) + \frac{\mu}{2}A(N)^4$$
$$+\frac{2}{3}(\nu + \mu\beta)A(N)^3 + (-\frac{1}{6}\mu\beta^2 + \frac{\beta\nu}{3} + \frac{\delta\mu}{2} + \alpha^2 + \epsilon)A(N)^2$$
$$+(-\frac{1}{6}\mu\beta\delta + \frac{\delta a}{3} + \alpha\gamma + 2\zeta)A(N)$$

and finally the structure function is

$$\Phi(N) =$$
$$\frac{\rho(N-1)^{-1}}{(\triangle A(N-1) - \frac{\beta}{2})(\beta^2 - \epsilon - 2\beta A(N) - \triangle A(N)^2) + (\triangle A(N) + \frac{\beta}{2})(\beta^2 - \delta - 2\beta A(N) - \triangle A(N-1)^2)}$$
$$((\triangle A(N) + \frac{\beta}{2})(K + 2\alpha A(N)^2 b(N) - (\beta^2 - \delta - 2\beta A(N))b(N)^2$$
$$-2(\alpha\beta - \gamma)A(N)b(N) - (\beta\gamma - 2\epsilon)b(N) - \frac{\mu}{2}A(N)^4 - \frac{2}{3}(\nu + \mu\beta)A(N)^3 \tag{2.22}$$
$$-(-\frac{1}{6}\mu\beta^2 + \frac{\beta\nu}{3} + \alpha^2 + \xi)A(N)^2 - (-\frac{1}{6}\mu\beta\delta + \frac{\delta\nu}{3} + \alpha\gamma + 2\zeta)A(N))$$
$$-\frac{1}{2}(\beta^2 - \delta - 2\beta A(N) - \triangle A(N)^2)(gA(N)^3 + \nu A(N)^2 - \beta b(N)^2 - 2\alpha A(N)b(N) + \xi A(N) - \gamma b(N) + \zeta))$$

Thus the structure function depends only on the function ρ. This function can be arbitrarily chosen and does not influence the spectrum. In Case 1 we choose

$$\rho(N) = \frac{1}{3 * 2^{12}\beta^8(N+u)(1+N+u)(1+2(N+u))^2} \tag{2.23}$$

From our expressions for A(N) , b(N) and $\rho(N)$, the third relation of the cubic associative algebra and the expression of the Casimir operator we find the structure function $\Phi(N)$. For the Case 1 the structure function is a polynomial of order 10 in N. The coefficients of this polynomial are functions of α, β, μ, γ, δ, ϵ, ν, ξ and ζ.

Case 2 $\beta = 0$ et $\delta \neq 0$

$$A(N) = \sqrt{\delta}(N+u), b(N) = -\alpha(N+u)^2 - \frac{\gamma}{\sqrt{\delta}}(N+u) - \frac{\epsilon}{\delta} \tag{2.24}$$

In Case 2 we choose a trivial expression $\rho(N) = 1$. The explicit expression of the structure function for this case is

$$\Phi(N) = (\frac{K}{-4\delta} - \frac{\gamma\epsilon}{4\delta^{\frac{3}{2}}} - \frac{\zeta}{4\sqrt{\delta}} + \frac{\epsilon^2}{4\delta^2}) \tag{2.25}$$

$$+(\frac{-\alpha\epsilon}{2\delta} - \frac{d}{4} - \frac{\gamma^2}{4\delta} + \frac{\gamma\epsilon}{2\delta^{\frac{3}{2}}} + \frac{\alpha\gamma}{4\sqrt{\delta}} + \frac{\zeta}{2\sqrt{\delta}} + \frac{\nu\sqrt{\delta}}{12})(N+u)$$

$$+(\frac{-\nu\sqrt{\delta}}{4} - \frac{3\alpha\gamma}{4\sqrt{\delta}} + \frac{\gamma^2}{4\delta} + \frac{\epsilon\alpha}{2\delta} + \frac{\alpha^2}{4} + \frac{\xi}{4} + \frac{\mu\delta}{8})(N+u)^2$$

$$+(\frac{-\alpha^2}{2} + \frac{\gamma\alpha}{2\delta^{\frac{1}{2}}} + \frac{\nu\sqrt{\delta}}{6} - \frac{\mu\delta}{4})(N+u)^3 + (\frac{\alpha^2}{4} + \frac{\mu\delta}{8})(N+u)^4$$

We will consider a representation of the cubic associative algebra in which the generator A and the Casimir operator K are diagonal. We use a parafermionic realization in which the parafermionic number operator N and the Casimir operator K are diagonal. The basis of this representation is the Fock basis for the parafermionic oscillator. The vector $|k, n>, n = 0, 1, 2...$ satisfies the following relations :

$$N|k,n>= n|k,n>, \quad K|k,n>= k|k,n> \tag{2.26}$$

The vectors $|k, n>$ are also eigenvectors of the generator A.

$$A|k,n>= A(k,n)|k,n>$$

$$A(k,n) = \frac{\beta}{2}((n+u)^2 - \frac{1}{4} - \frac{\delta}{\beta^2}), \beta \neq 0 \tag{2.27}$$

$$A(k,n) = \sqrt{\delta}(n+u), \beta = 0, \delta \neq 0$$

We have the following constraints for the structure function,

$$\Phi(0,u,k) = 0, \quad \Phi(p+1,u,k) = 0 \tag{2.28}$$

With these two relations we can find the energy spectrum. Many solutions for the system exist. Unitary representations of the deformed parafermionic oscillator obey the following constraint $\Phi(x) > 0$ for x=1,2,...,p .

3 Examples

There exist 21 quantum potentials separable in cartesian coordinates with a third order integral, we will consider one interesting case in which the cubic algebra allows us to calculate the energy spectrum.

Case Q5

$$H = \frac{P_x^2}{2} + \frac{P_y^2}{2} + \hbar^2\left(\frac{x^2+y^2}{8a^4} + \frac{1}{(x-a)^2} + \frac{1}{(x+a)^2}\right) \tag{3.1}$$

$$A = \frac{P_x^2}{2} - \frac{P_y^2}{2} + \hbar^2\left(\frac{x^2-y^2}{8a^4} + \frac{1}{(x-a)^2} + \frac{1}{(x+a)^2}\right) \tag{3.2}$$

$$\begin{aligned} B = X_2 = \{L, P_x^2\} + \hbar^2\{y(\frac{4a^2-x^2}{4a^4} - \frac{6(x^2+a^2)}{(x^2-a^2)^2)}), P_x\} \\ +\hbar^2\{x(\frac{(x^2-4a^2)}{4a^4} - \frac{2}{x^2-a^2} + \frac{4(x^2+a^2)}{(x^2-a^2)^2}), P_y\} \end{aligned} \tag{3.3}$$

The integrals A,B and H give rise to the algebra

$$\begin{aligned} [A,B] &= C \\ [A,C] &= \frac{h^4}{a^4}B \\ [B,C] &= -32\hbar^2A^3 - 48\hbar^2A^2H + 16\hbar^2H^3 + 48\frac{\hbar^4}{a^2}A^2 + 32\frac{\hbar^4}{a^2}HA - 16\frac{\hbar^4}{a^2}H^2 \\ &\quad +8\frac{\hbar^6}{a^4}A - 4\frac{\hbar^6}{a^4}H - 12\frac{\hbar^8}{a^6} \quad . \end{aligned} \tag{3.4}$$

The Casimir operator is

$$K = -16\hbar^2H^4 + 32\frac{\hbar^4}{a^2}H^3 + 16\frac{\hbar^6}{a^4}H^2 - 40\frac{\hbar^8}{a^6}H - 3\frac{\hbar^{10}}{a^8} \quad . \tag{3.5}$$

and we have

$$\Phi(x) = (4\frac{a^4}{\hbar^2}H^4 - 12a^2H^3 + 11\frac{\hbar^4}{a^2}H - \frac{15}{4}\frac{h^6}{a^4}) \tag{3.6}$$
$$+(8a^2H^3 - 8\hbar^2H^2 - 14\frac{\hbar^4}{a^2}H - 4\frac{\hbar^6}{a^4})(x+u) + (20\frac{\hbar^4}{a^2}H - 14\frac{\hbar^6}{a^4})(x+u)^2$$
$$+(-8\frac{\hbar^4}{a^2}H + 16\frac{\hbar^6}{a^4})(x+u)^3 - 4\frac{\hbar^8}{a^4}(x+u)^4$$

$$\Phi(x) = (\frac{-4\hbar^8}{a^4})(x+u-(\frac{-a^2E}{\hbar^2}-\frac{1}{2}))(x+u-(\frac{a^2E}{\hbar^2}+\frac{1}{2}))(x+u-(\frac{-a^2E}{\hbar^2}+\frac{3}{2}))(x+u-(\frac{-a^2E}{\hbar^2}+\frac{5}{2})) \tag{3.7}$$

We find u with

$$\Phi(0,u,k) = 0$$

$$u_1 = \frac{-a^2E}{\hbar^2} - \frac{1}{2}, \quad u_2 = \frac{a^2E}{\hbar^2} + \frac{1}{2}, \quad u_3 = \frac{-a^2E}{\hbar^2} + \frac{3}{2}, \quad u_4 = \frac{-a^2E}{\hbar^2} + \frac{5}{2} \tag{3.8}$$

We have four cases :
Case 1 $u = u_1$

$$E = \frac{\hbar^2 p}{2a^2}, \quad \Phi(x) = (\frac{4\hbar^8}{a^4})x(p+1-x)(x-2)(x-3) \quad . \tag{3.9}$$

Case 2 $u = u_2$

$$E = \frac{-\hbar^2(p+2)}{2a^2}, \quad \Phi(x) = (\frac{4\hbar^8}{a^4})x(p+1-x)(p+3-x)(p+4-x) \tag{3.10}$$

$$E = \frac{-\hbar^2 p}{2a^2}, \quad \Phi(x) = (\frac{4\hbar^8}{a^4})x(p+1-x)(p-1-x)(p-2-x) \tag{3.11}$$

$$E = \frac{-\hbar^2(p-1)}{2a^2}, \quad \Phi(x) = (\frac{4\hbar^8}{a^4})x(p+1-x)(p-2-x)(p-x) \quad . \tag{3.12}$$

Case 3 $u = u_3$

$$E = \frac{\hbar^2(p+2)}{2a^2}, \quad \Phi(x) = (\frac{4\hbar^8}{a^4})x(p+1-x)(x-1)(x+2) \quad . \tag{3.13}$$

Case 4 $u = u_4$

$$E = \frac{\hbar^2(p+3)}{2a^2}, \quad \Phi(x) = (\frac{4\hbar^8}{a^4})x(p+1-x)(x+1)(x+3) \quad . \tag{3.14}$$

The only case that correspond to unitary representations are (3.10) and (3.14).

4 Conclusion

The main results of this article are that we have constructed the associative algebras for superintegrable potential with a second order integral and a third order integral. We find realizations in terms of deformed oscillator algebras for the cubic associative algebras. We apply our result to a speficic potential[13,14]. We leave the other quantum cases to a future article.
We see that many systems in classical and quantum physics are described by a nonlinear symmetry that provides information about the energy spectrum.
We note that our polynomial algebras and their realizations are independant of the choice of coordinate system. We could apply our results in the future to systems with a third order integral that are separable in polar, elliptic or parabolic coordinates. The method is independant of the metric and we could apply our polynomial algebras to other cases than superintegrable potentials in E(2).

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